# NONLINEAR ANISOTROPIC ELASTIC PROPERTIES OF THE CANINE AORTA

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ABSTRACT A nonlinear theory of large elastic deformations of the aortic tissue has been developed. The wall tissue has been considered to be incompressible and curvilinearly orthotropic. The strain energy density function for the tissue is expressed as a polynomial in the circumferential and longitudinal Green-St. Venant strains. Limiting application to states of strains wherein the geometric axes are the principal axes and truncating the energy expression to include terms with highest degrees 2, 3, and 4, three expressions with 3, 7, and 12 constitutive constants are obtained. Results of application of these expressions to data from three series of in vitro and in vivo experiments involving 31 dogs have been presented. Whereas all the three expressions are found to be applicable to various degrees, the third-degree expression for the strain energy density function with seven constitutive constants is particularly recommended for general use.

## INTRODUCTION

In many physiologic considerations such as the study of pulse-wave velocity in blood vessels, design of prosthetics and artificial organs, and investigation of vascular damage under various situations, it is important to know the relation between forces acting on a blood vessel and the resulting deformation. During the last decade this problem of characterization of mechanical properties of blood vessels has been a subject of intensive investigation as cited by Fung (1968) and Patel and Vaishnav (1972).

The blood vessel tissue is capable of undergoing large deformations, and hence, the classical linear elasticity theory cannot be applied to it. Moreover, the tissue is anisotropic, i.e., it exhibits different material responses in the circumferential, longitudinal, and radial directions. Finally, the blood vessel tissue is incompressible. No previous work simultaneously takes into account all these aspects of tissue behavior. Patel et al. (1969) evaluated orthotropic *incremental* elastic properties of the canine aorta at various circumferential and longitudinal stretches. Whereas the incremental approach provides useful information, it is limited by the fact that comparisons can be made among different tissues only around identical states of strain. The nonlinear theory of Tickner and Sacks (1964, 1967) suffers from inadvertent use of a formulation valid only for isotropic materials. Finally, the work of Simon et al. (1971) properly starts with a formulation valid for an orthotropic, nonlinearly elastic, incompressible material, but the theory is developed only for experiments involving inflation at constant longitudinal stretch.

In the present work, therefore, we (*a*) present a theory for characterizing the mechanical response of a canine aorta treating it as composed of an incompressible, curvilinearly orthotropic material capable of undergoing large deformations, (*b*) apply it to data obtained from canine arteries subjected to various circumferential and longitudinal stretches, and (*c*) obtain a consistent set of constitutive material constants applicable in the experimental range of stretches. We give below a brief theoretical development of the proposed constitutive relation. For a more detailed derivation, reference is made to Patel and Vaishnav (1972).

## THEORETICAL DEVELOPMENT

The purpose of the present treatment is to develop a constitutive relation between stresses in an aortic segment and the associated deformation. We shall make, at the outset, a number of assumptions which are permissible for many physiological considerations (Wolinsky and Glagov, 1964, 1967; Carew et al., 1968; Patel and Fry, 1969). We shall assume that a short (about 8 cm long) segment of the canine middle descending thoracic aorta can be considered to be a thin-walled cylindrical tube of circular cross-section and uniform thickness, and further, that the aortic tissue is homogeneous, incompressible, elastic, and cylindrically orthotropic. For a detailed discussion of these assumptions including experimental verification we refer to Patel and Vaishnav (1972).

Let us now consider an arterial segment in its undeformed cylindrical configuration. Let this configuration be referred to a cylindrical coordinate system[1] $\theta'^i$ ($\theta'^1 \equiv \rho, \theta'^2 \equiv \vartheta, \theta'^3 \equiv x_3 \equiv z$) with the $\rho$ axis directed radially, the $\vartheta$ coordinate line circumferentially, and the $x_3$ axis along the geometric axis of the segment. Under the action of external forces, the segment will assume a deformed configuration which we refer to another cylindrical system $\theta^i$ ($\theta^1 \equiv r, \theta^2 \equiv \theta, \theta^3 \equiv y_3$). The deformed configuration can be completely defined by prescribing the position coordinates $\theta^i$ in the deformed configuration as functions of the material coordinate $\theta'^j$ of all particles in the undeformed configuration.

The square of the distance between any two neighboring material points with coordinates $\theta'^i$ and $\theta'^i + d\theta'^i$ in the undeformed segment is given by

$$ds_0^2 = g'_{ij}\, d\theta'^i\, d\theta'^j, \tag{1}$$

[1] We follow here the approach and notation of Green and Adkins (1960) as far as possible.

where

$$g'_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

is the metric tensor of the coordinate system $\theta'^i$.

Similarly, the squared distance between the corresponding points in the deformed configuration with coordinates $\theta^i$ and $\theta^i + d\theta^i$ is given by

$$ds^2 = G_{ij}\, d\theta^i\, d\theta^j = G'_{ij}\, d\theta'^i\, d\theta'^j, \tag{3}$$

where

$$G_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

is the metric tensor of the coordinate system $\theta^i$ and $G'_{ij}$ is given by

$$G'_{ij} = \frac{\partial\theta^p}{\partial\theta'^i}\frac{\partial\theta^q}{\partial\theta'^j} G_{pq}\,. \tag{5}$$

The strain in the body in the deformed configuration relative to the undeformed configuration is obtained by computing the increase in the squared length $ds_0^2$ as

$$ds^2 - ds_0^2 = 2\gamma'_{ij}\, d\theta'^i\, d\theta'^j, \tag{6}$$

where

$$\gamma'_{ij} = \tfrac{1}{2}(G'_{ij} - g'_{ij}), \tag{7}$$

is called the Green-St. Venant strain tensor.

We also introduce the physical components $\gamma'_{(ij)}$ of $\gamma'_{ij}$ defined as

$$\gamma'_{(ij)} = \gamma'_{ij}/\sqrt{(g'_{ii}g'_{jj})} \qquad \text{(no sum on } i \text{ and } j). \tag{8}$$

The physical components correspond to the values of $\gamma'_{ij}$ in a local cartesian coordinate system with axes oriented along the local $\theta'^i$ axes. We assume that the axes of curvilinear anisotropy for the artery coincide with the $\theta'^i$ axes.

If $\tau^{ij}$ denote the contravariant components of the stress tensor in the coordinate system $\theta^i$, then the problem of finding the elastic constitutive relation for the arterial tissue reduces to that of finding an explicit formulation of $\tau^{ij}$ as functions of $\gamma'_{(kl)}$. There are two possible approaches to the problem. The first approach, due to Cauchy, is to express $\tau^{ij}$ directly as functions of $\gamma'_{(kl)}$. The second approach, due to Green, is to assume the existence of a strain energy density function $W$ per unit initial volume, finding the explicit form of $W = W(\gamma'_{(ij)})$ and computing $\tau^{ij}$ therefrom using the expression

$$\tau^{ij} = p' G_{ij} + \tfrac{1}{2}[A^{ij}_{(rs)} + A^{ij}_{(sr)}] \frac{\partial W}{\partial \gamma'_{(rs)}} \qquad \text{(sum on } r \text{ and } s\text{)}, \tag{9}$$

where

$$A^{ij}_{(rs)} = \frac{1}{\sqrt{(g'_{rr} g'_{ss})}} \frac{\partial \theta^i}{\partial \theta'^r} \frac{\partial \theta^j}{\partial \theta'^s} \qquad \text{(no sum on } r \text{ and } s\text{)}, \tag{10}$$

and $p'$ denotes an arbitrary hydrostatic pressure not determinable from the knowledge of strain only as a result of the incompressibility of the arterial tissue.

The assumption of the existence of a strain energy density function provides a more economical representation of elastic properties. At least for isothermal and adiabatic deformations the existence of a strain energy density function is provable. Following customary practice, we shall assume the existence of a strain energy density function for the general case also.

As the arterial tissue is incompressible (Carew et al., 1968) and cylindrically orthotropic (Patel and Fry, 1969), it can be proved that its strain energy density function must be of the form

$$W = W(\gamma'_{(11)}, \gamma'_{(22)}, \gamma'_{(33)}, \gamma'^2_{(12)}, \gamma'^2_{(13)}, \gamma'^2_{(23)}), \tag{11}$$

or

$$W = W(a, b, c, d^2, e^2, f^2), \tag{12}$$

where

$$\gamma'_{(11)} = c, \gamma'_{(22)} = a, \gamma'_{(33)} = b, \gamma'_{(12)} = d, \gamma'_{(13)} = e, \gamma'_{(23)} = f. \tag{13}$$

We shall assume that $W$ can be approximated as closely as desired by a polynomial in its arguments. As the stresses depend only on the derivatives of $W$, the polynomial need not have a constant term. Further, it can be proved that the coefficients of the first-degree terms can be set equal to zero if the natural state of the material ($a = b = c = d = e = f = 0$) is assumed to be stress-free. Thus, $W$ can be expressed as

$$W = k_1 a^2 + k_2 b^2 + k_3 c^2 + k_4 d^2 + k_5 e^2 + k_6 f^2 + k_7 ab + k_8 bc + k_9 ac + \cdots \tag{14}$$

For practical considerations the unending polynomial must be truncated somewhere. In general, the larger the number of terms retained, the better the accuracy of representation of material response over a given range of arguments of $W$. The larger the number of terms retained, however, the larger the experimental effort necessary for evaluation of the material constants and the more the noise in the obtained values of these constants. Thus, retaining too many terms may not only be unnecessary but unjustified also.

We can considerably reduce the number of terms by restricting application of the theory to "physiologic" states of strain. We use the term "physiologic" to connote a state of strain in the blood vessel segment arising from application of intramural pressure and a uniform longitudinal force. By virtue of orthotropy, under this type of loading, no shearing strains will develop, i.e., $d = e = f = 0$. Therefore, in the polynomial expression for $W$, the terms containing $d$, $e$, and $f$ can be neglected.

Further simplification in the polynomial representation for $W$ can be obtained by utilizing the fact that for an incompressible material, only isochoric (volume-preserving) deformations are admissible. Thus, $a$, $b$, and $c$ are interrelated by the constraint

$$(1 + 2a)(1 + 2b)(1 + 2c) = 1, \tag{15}$$

when $d = e = f = 0$. Equation 15 follows from the fact that the quantities $(1 + 2a)$, $(1 + 2b)$, and $(1 + 2c)$, which, respectively, equal the squares of the stretches $\lambda_\theta$, $\lambda_z$, and $\lambda_r$ in the $\theta$, $z$, and $r$ directions, become squares of the principal stretches when $d = e = f = 0$, and the product of the principal stretches provides the new volume of a cube of unit initial dimensions.

Equation 15 can be used to express $c$ as a power series in $a$ and $b$ and by truncating the series and substituting for $c$ in equation 14, $c$ can be eliminated from the expression for $W$. After recombining terms of various degrees in $a$ and $b$, we get, from equation 14, the following second, third, and fourth degree expressions for $W$:

$$W = A_1a^2 + B_1ab + C_1b^2, \tag{16}$$

$$W = A_2a^2 + B_2ab + C_2b^2 + D_2a^3 + E_2a^2b + F_2ab^2 + G_2b^3, \tag{17}$$

$$W = A_3a^2 + B_3ab + C_3b^2 + D_3a^3 + E_3a^2b + F_3ab^2 + G_3b^3 + H_3a^4 + I_3a^3b + J_3a^2b^2 + K_3ab^3 + L_3b^4. \tag{18}$$

Expressions of higher degree can be written down in a similar manner if desired.

We shall refer to the theories postulating the expressions for $W$ of degrees 2, 3, and 4 as the 3-constant, 7-constant, and the 12-constant theories, respectively. We have distinctly suffixed the constitutive constants from the three theories to emphasize the fact that when more than one theory is used on the same set of data, the corresponding constants do not have to be equal. Suffixes will be omitted when no confusion is likely to occur.

Now, for a physiologic type of state of strain, the deformation $\theta^i = \theta^i(\theta'^j)$ is given by

$$\rho = \rho(r), \qquad \vartheta = \theta, \qquad x_3 = y_3/\lambda_3, \tag{19}$$

or,

$$r = r(\rho), \qquad \theta = \vartheta, \qquad y_3 = \lambda_3 x_3. \tag{20}$$

For such a deformation we have

$$\frac{\partial\theta^i}{\partial\theta'^j} = \begin{bmatrix} \partial r/\partial\rho & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \tag{21}$$

$$G_{ij} = \begin{bmatrix} (\partial r/\partial\rho)^2 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \tag{22}$$

and

$$\gamma'_{(ij)} = \begin{bmatrix} \tfrac{1}{2}\{(\partial r/\partial\rho)^2 - 1\} & 0 & 0 \\ 0 & \tfrac{1}{2}(r^2/\rho^2 - 1) & 0 \\ 0 & 0 & \tfrac{1}{2}(\lambda_3^2 - 1) \end{bmatrix}. \tag{23}$$

From equations 19, 4, 10, 2, 21, and 18 we get the following expressions for the stresses:

$$S_r \equiv \tau^{11} = p', \tag{24}$$

$$\begin{aligned} S_\theta \equiv r^2\tau^{22} &= p' + 2Aa + Bb + (4A + 3D)a^2 + (2B + 2E)ab + Fb^2 \\ &\quad + (6D + 4H)a^3 + (4E + 3I)a^2b + (2F + 2J)ab^2 + Kb^3 \\ &\quad + 8Ha^4 + 6Ia^3b + 4Ja^2b^2 + 2Kab^3, \\ &= p' + A(2a + 4a^2) + B(b + 2ab) + D(3a^2 + 6a^3) \\ &\quad + E(2ab + 4a^2b) + F(b^2 + 2ab^2) + H(4a^3 + 8a^4) \\ &\quad + I(3a^2b + 6a^3b) + J(2ab^2 + 4a^2b^2) + K(b^3 + 2ab^3), \end{aligned} \tag{25}$$

$$\begin{aligned} S_z \equiv \tau^{33} &= p' + Ba + 2Cb + Ea^2 + (2B + 2F)ab + (4C + 3G)b^2 + Ia^3 \\ &\quad + (2E + 2J)a^2b + (4F + 3K)ab^2 + (6G + 4L)b^3 + 2Ia^3b \\ &\quad + 4Ja^2b^2 + 6Kab^3 + 8Lb^4, \\ &= p' + B(a + 2ab) + C(2b + 4b^2) + E(a^2 + 2a^2b) \\ &\quad + F(2ab + 4ab^2) + G(3b^2 + 6b^3) + I(a^3 + 2a^3b) \\ &\quad + J(2a^2b + 4a^2b^2) + K(3ab^2 + 6ab^3) + L(4b^3 + 8b^4). \end{aligned} \tag{26}$$

Equations 24–26 are the stress-strain relations for the aortic tissue under physiologic loading. The quantities $S_r$, $S_\theta$, and $S_z$ are the actual stresses (physical components of $\tau^{ij}$) on the area elements perpendicular to the radial, circumferential, and the longitudinal directions, respectively, and are normal to the planes of the area elements. We observe that only the stress differences $S_\theta - S_r$ and $S_z - S_r$ are determinable from the knowledge of the constitutive constants and the strains. The stress-strain relations for the 7- and 3-constant theories can be obtained from equations 24–26 by setting appropriate constants equal to zero.

*Derivation of the Incremental Theory from the General Nonlinear Theory*

Often it is required to know the incremental behavior of a blood vessel (Biot, 1965; Patel et al., 1969; Patel and Vaishnav, 1972) whose over-all response to known. The relations necessary for this can be derived in an easy manner by computing the differentials of the stress differences $S_\theta - S_r$ and $S_z - S_r$ and using the relations

$$e_\theta = \frac{d\lambda_\theta}{\lambda_\theta}, \quad e_z = \frac{d\lambda_z}{\lambda_z}, \quad e_r = \frac{d\lambda_r}{\lambda_r}; \tag{27}$$

$$\lambda_\theta^2 = 1 + 2a, \quad \lambda_z^2 = 1 + 2b, \quad \lambda_r\lambda_\theta\lambda_z = 1; \tag{28}$$

$$da = (1 + 2a)e_\theta, \quad db = (1 + 2b)e_z; \tag{29}$$

$$e_r + e_\theta + e_z = 0; \tag{30}$$

where $\lambda_\theta$, $\lambda_z$, and $\lambda_r$ are the stretches in the circumferential, longitudinal, and radial directions, respectively, and $e_\theta$, $e_z$, and $e_r$ are the corresponding incremental strains.

Thus, we have from equations 24–26, the following relations for incremental stresses in terms of incremental strains:

$$P_\theta - P_r = \beta_{\theta\theta}e_\theta + \beta_{\theta z}e_z, \tag{31}$$

$$P_z - P_r = \beta_{z\theta}e_\theta + \beta_{zz}e_z, \tag{32}$$

where

$$P_\theta \equiv dS_\theta, \quad P_z \equiv dS_z, \quad P_r \equiv dS_r; \tag{33}$$

$$\beta_{\theta\theta} = (1 + 2a)[A(2 + 8a) + 2Bb + D(6a + 18a^2) + E(2b + 8ab) + 2Fb^2 + H(12a^2 + 32a^3) + I(6ab + 18a^2b) + J(2b^2 + 8ab^2) + 2Kb^3]; \tag{34}$$

$$\beta_{\theta z} = \beta_{z\theta} = (1 + 2a)(1 + 2b)(B + 2Ea + 2Fb + 3Ia^2 + 4Jab + 3Kb^2); \tag{35}$$

$$\beta_{zz} = (1 + 2b)[2Ba + C(2 + 8b) + 2Ea^2 + F(2a + 8ab) + G(6b + 18b^2) + 2Ia^3 + J(2a^2 + 8a^2b) + K(6ab + 18ab^2) + L(12b^2 + 32b^3)]. \tag{36}$$

The inverse relations expressing the incremental strains in terms of incremental stresses may be written in the following two useful forms:

$$e_r = C_{rr}P_r + C_{r\theta}P_\theta + C_{rz}P_z = \frac{1}{E_r} P_r - \frac{\sigma_{r\theta}}{E_\theta} P_\theta - \frac{\sigma_{rz}}{E_z} P_z , \tag{37}$$

$$e_\theta = C_{\theta r}P_r + C_{\theta\theta}P_\theta + C_{\theta z}P_z = -\frac{\sigma_{\theta r}}{E_r} P_r + \frac{1}{E_\theta} P_\theta - \frac{\sigma_{\theta z}}{E_z} P_z , \tag{38}$$

$$e_z = C_{zr}P_r + C_{z\theta}P_\theta + C_{zz}P_z = -\frac{\sigma_{zr}}{E_r} P_r - \frac{\sigma_{z\theta}}{E_\theta} P_\theta + \frac{1}{E_z} P_z , \tag{39}$$

where $E_r$, $E_\theta$, $E_z$ are the orthotropic Young's moduli and $\sigma_{r\theta}$, $\sigma_{rz}$, etc., are the orthotropic Poisson's ratios. The following formulas express various constants appearing in equations 37–39 in terms of $\beta_{\theta\theta}$, $\beta_{\theta z}$, and $\beta_{zz}$, and hence, to $A$, $B$, $\cdots$, $L$, through equations 34–36:

$$C_{rr} = \frac{1}{\beta}(\beta_{zz} + \beta_{\theta\theta} - 2\beta_{\theta z}), \qquad C_{\theta\theta} = \beta_{zz}/\beta, \qquad C_{zz} = \beta_{\theta\theta}/\beta; \tag{40}$$

$$C_{r\theta} = C_{\theta r} = (\beta_{\theta z} - \beta_{zz})/\beta; \tag{41}$$

$$C_{zr} = C_{rz} = (\beta_{\theta z} - \beta_{\theta\theta})/\beta; \tag{42}$$

$$C_{\theta z} = C_{z\theta} = -\beta_{\theta z}/\beta; \tag{43}$$

$$\beta = \beta_{\theta\theta}\beta_{zz} - \beta_{\theta z}^2 ; \tag{44}$$

$$E_r = \beta/(\beta_{\theta\theta} + \beta_{zz} - 2\beta_{\theta z}), \qquad E_\theta = \beta/\beta_{zz} , \qquad E_z = \beta/\beta_{\theta\theta} ; \tag{45}$$

$$\sigma_{r\theta} = (\beta_{zz} - \beta_{\theta z})/\beta_{zz} , \qquad \sigma_{\theta r} = (\beta_{zz} - \beta_{\theta z})/(\beta_{\theta\theta} + \beta_{zz} - 2\beta_{\theta z}); \tag{46}$$

$$\sigma_{\theta z} = \beta_{\theta z}/\beta_{\theta\theta} , \qquad \sigma_{z\theta} = \beta_{\theta z}/\beta_{zz} ; \tag{47}$$

$$\sigma_{zr} = (\beta_{\theta\theta} - \beta_{\theta z})/(\beta_{\theta\theta} + \beta_{zz} - 2\beta_{\theta z}), \qquad \sigma_{rz} = (\beta_{\theta\theta} - \beta_{\theta z})/\beta_{\theta\theta} . \tag{48}$$

The relations

$$\sigma_{r\theta} + \sigma_{z\theta} = \sigma_{rz} + \sigma_{\theta z} = \sigma_{\theta r} + \sigma_{zr} = 1, \tag{49}$$

easily derivable from equations 46–48, are noteworthy.

It should be noted that nowhere in the preceding development the assumption of thinness of vessel wall was invoked. It is only in the application of the theory that such an assumption will be used for convenience.

*Relations among Pressure, Longitudinal Force, and the Stresses in the Wall*

Now suppose that the arterial segment has mean radius $R_0$, thickness $h_0$, and length $L_0$ in the undeformed configuration. Under the action of intramural pressure $p$

and longitudinal force $F$, let the segment assume the values $R$, $h$, and $L$, for the mean radius, thickness, and length, respectively. From the requirement of incompressibility, we see that a thin-walled cylindrical layer of the artery must preserve its volume. Thus, if the layer has initial internal radius $\rho$, thickness $d\rho$, and length $L_0$, and final radius $r$, thickness $dr$, and length $L = \lambda_z L_0$, then we must have $2\pi\rho \, d\rho L_0 = 2\pi r \, dr \lambda_z L_0$, which upon integration gives

$$\rho^2 = \lambda_z r^2 + k, \tag{50}$$

where $k$ is a constant of integration. Thus, $k$ and $\lambda_z$ are the only geometrical parameters required to define completely the deformed configuration if the undeformed dimensions are known.

Through integration of the equations of equilibrium, it can be proved that (Green and Adkins, 1960) the radial stress $S_r$ is given by

$$S_r = -\int_r^{R+h/2} (S_\theta - S_r) \frac{dr}{r}, \tag{51}$$

if the extramural pressure is zero. Also, the intramural pressure $p$ and the longitudinal force $F$ must obey the following relations:

$$p = \int_{R-h/2}^{R+h/2} (S_\theta - S_r) \frac{dr}{r}, \tag{52}$$

and

$$F = -p\pi\left(R - \frac{h}{2}\right)^2 + 2\pi \int_{R-h/2}^{R+h/2} S_z r \, dr. \tag{53}$$

If the deformed configuration is known, it is possible to compute $a$, which is a function of $r$, and $b$, which we will assume to be constant. If the constitutive constants $A, \cdots, L$ are known, the stress difference $S_\theta - S_r$ is then computed using equation 25, $S_r$ using equation 51, and $S_z$ using equation 26. The values of $p$ and $F$ required to maintain the deformed configuration are then known. Both the reverse problem of finding $k$ and $\lambda_z$ for given $p$ and $F$ and the problem of finding $A, B, \cdots, L$ from a known set of corresponding values of $p$, $F$, $k$, and $\lambda_z$ are difficult and require iterative methods of solution. We are concerned here with the latter problem of computing the constitutive constants and resort to the assumption of thinness of the vessel wall for expediency. Accordingly, we assign the values of $a$ and $b$ at $r = R$ to all points through the thickness. $(S_\theta - S_r)$ and $(S_z - S_r)$ then do not depend on $r$. Consequently, we have, from equations 51–53, after a few simplifications,

$$S_r = -p/2, \tag{54}$$

$$S_\theta = p(R/h - \tfrac{1}{2}), \tag{55}$$

$$S_z = \frac{p}{2}(R/h - 1) + F/2\pi Rh. \qquad (56)$$

The value of $S_r$ in equation 54 is taken to be a constant equal to the mean of the values at the inner and outer surfaces.

Below we describe the methods used in carrying out a series of experiments to test the applicability of the 3-, 7-, and 12- constant theories. The experiments basically consisted of subjecting canine aortic segments to various levels of $p$ and $F$, measuring the corresponding $R + h/2$ and $L$ at each level, and measuring $R_0$, $L_0$, and $h_0$ at the end of the experiment.

## METHODS

13 mongrel dogs weighing 21.4–33.6 kg (average weight 26.3 kg) were studied under sodium pentobarbital (Nembutal, Abbott Laboratories, Chemical Marketing Div., North Chicago, Ill.) anesthesia (about 0.65 cc/kg). The middle descending thoracic aorta was exposed and a relatively uniform segment, approximately 8 cm in length,[2] was marked. Its range of external radii was measured with vernier calipers and two plastic cylindrical plugs were selected to fit snugly into the lumen at each end of the vessel segment. The plugs were then introduced into the vessel lumen and coupled to the vessel wall by tying sutures externally around the vessel over the circumferential grooves in the plugs (Patel et al., 1969; Patel and Fry, 1969). The segment was removed and stripped of its surrounding tissues up to the adventitia.

The experimental setup was very similar to that used by Patel and Janicki (1970). A hollow metal rod was screwed into a threaded hole through the center of the proximal plug so that the intravascular pressure could be changed; a solid metal plug containing a hook was screwed into the distal plug so that weights could be hung from it; and the vessel segment was mounted vertically on a ring stand so that its upper end was fixed. The hollow metal rod connected the vessel to a reservoir filled with oxygenated blood kept at room temperature (26–28°C). The reservoir was made airtight and connected to a source of air pressure which could be adjusted by using a bleeder valve. This pressure was monitored by a P23 Db Statham transducer (Statham Instruments, Inc., Los Angeles, Calif.) whose zero pressure reference was set at the midsegment level. The length $L$ (i.e., the distance between the ties coupling the vessel to the plugs) was measured using a millimeter scale which was mounted to the stand behind the segment, and the midsegment external diameter $D$ was measured using vernier calipers.

The experimental procedure consisted of first removing the effects of hysteresis by subjecting the vessel segment to two cycles of inflation and deflation over a range of 20–200 cm $H_2O$ (Patel et al., 1969). The vessel was then longitudinally loaded with a known weight and the pressure was varied over a range of 25–200 cm $H_2O$ in steps of 25 cm $H_2O$. The length and diameter were measured at each of these pressure levels 1 min after the pressure step was imposed; this time was considered adequate for further increase in strain due to creep to be negligible (Patel et al., 1969). The pressure was then returned to its 25 cm $H_2O$ level, the longitudinal load increased to a new value, and the range of pressures repeated. Altogether five longitudinal loads were used: 0, 25, 50, 75, and 100 gms. Throughout the experiment the vessel was kept moist, externally, by dripping saline (at room temperature) over it. At the end of the experiment the blood vessel was slit open longitudinally, and the unstressed length

[2] This length was considered adequate to avoid "end effects" and short enough to trap a reasonably uniform aortic segment, minimizing taper.

$L_0$ and circumference were measured. The value of the unstressed radius $R_0$ was obtained by dividing the value of the unstressed circumference by $2\pi$. The wall volume $V$ was then determined by measuring its loss of weight when suspended in distilled water. Since the material is incompressible (Carew et al., 1968), $V$ is a constant and the values of thickness $h$ could be calculated for different values of D and $L$ from

$$h = D/2 - (D^2/4 - V/\pi L)^{1/2}. \quad (57)$$

The midwall radius $R$ was then calculated by subtracting $h/2$ from the measured external radius.

The material constants were subsequently calculated. First, the midwall circumferential and longitudinal stretches, $\lambda_\theta$ and $\lambda_z$, respectively, were calculated as

$$\lambda_\theta = \frac{R}{R_0}, \quad (58)$$

$$\lambda_z = \frac{L}{L_0}, \quad (59)$$

and the corresponding Green-St. Venant strains, $a$ and $b$, were computed for a given combination of pressure and weight using equation 28; the corresponding stresses $S_\theta$, $S_z$, and $S_r$, were calculated using equations 54–56. The knowledge of stress differences, $S_\theta - S_r$ and $S_z - S_r$, for each pair of values of $a$ and $b$, gave, using equations 24–26, two simultaneous equations with the 3, 7, or 12 constitutive constants as unknowns. Thus, for the 40 experimental points an overdetermined system with 80 simultaneous equations was obtained for each case, which was solved by the method of least squares. Three sets of constants were obtained corresponding to the 3-, 7-, and 12-constant theories.

The experiments covered an average range in $\lambda_\theta$ from 1.12 to 1.82 and in $\lambda_z$ from 1.17 to 1.60. The average maximum values of $S_\theta - S_r$ and $S_z - S_r$ were $3.56 \times 10^6$ and $2.23 \times 10^6$ dynes/cm$^2$, respectively.

## RESULTS AND DISCUSSION

Table I shows the average values ($\pm$SEM) of the constitutive constants for the 3-, 7-, and 12-constant theories. Whereas the 7- and 12-constant theories are too complex for direct interpretation, the following observations can be made from the coefficients of the 3-constant theory:

(*a*) The SEM values are very small indicating a basic quantitative similarity of the over-all stress-strain behavior among the 13 dogs.

(*b*) A glance at equation 16 shows that the constant $A_1$ controls the direct effect of the circumferential strain on the strain energy density $W$ the constant $C_1$ controls the direct effect of the longitudinal strain on $W$, and the constant $B_1$ denotes the interaction effects of the two strains on $W$. The fact that $A_1 > C_1$ thus indicates that the blood vessel is relatively stiffer in the circumferential direction than in the longitudinal direction.

(*c*) It is also interesting to note that the constants $A_1$ and $C_1$ from the 3-constant theory match fairly well the constants $A_2$ and $C_2$ from the 7-constant theory.

TABLE I

AVERAGE CONSTITUTIVE CONSTANTS (±SEM) FROM IN VITRO EXPERIMENTS ON 13 DOGS

| Theory | Constitutive constants |
|---|---|
| | *dynes/cm² × 10³* |
| 3-Constant theory | $A_1 = 332 \pm 11$  $B_1 = 315 \pm 18$  $C_1 = 252 \pm 14$ |
| 7-Constant theory | $A_2 = 323 \pm 19$  $B_2 = 34 \pm 54$  $C_2 = 247 \pm 20$  $D_2 = 25 \pm 12$  $E_2 = 68 \pm 28$  $F_2 = 862 \pm 411$  $G_2 = -41 \pm 32$ |
| 12-Constant theory | $A_3 = 528 \pm 53$  $B_3 = -114 \pm 107$  $C_3 = 462 \pm 87$  $D_3 = -138 \pm 80$  $E_3 = -391 \pm 212$  $F_3 = 862 \pm 411$  $G_3 = -600 \pm 300$  $H_3 = 136 \pm 36$  $I_3 = -338 \pm 133$  $J_3 = 1055 \pm 452$  $K_3 = -1157 \pm 589$  $L_3 = 623 \pm 35$ |

Minimum $\lambda_\theta$ = 1.12 ±0.01; maximum $\lambda_\theta$ = 1.82 ±0.02; minimum $\lambda_z$ = 1.17 ±0.01; maximum $\lambda_z$ = 1.61 ±0.02.

To test the fidelity with which the constitutive constants for each dog from each of the three theories fit the original data, the stress differences, $S_\theta - S_r$ and $S_z - S_r$, were computed for each dog at each state of strain using equations 24–26 and compared with the values of the computed stress differences as ordinates $y$ vs. the measured stress differences as abscissa $x$. These stress differences should ideally give a straight line with unit slope, zero intercept on the $y$ axis, and unit correlation coefficient. The average values for the correlation coefficient, slope, and intercept obtained using a least squares linear regression formula are given in Table II. Also given in Table II are values of the average per cent error obtained as the average of the absolute values of the differences between the computed and the measured stress differences and expressed as percentages of the measured stress differences.

Considerable information about the relative merits of the three theories can be obtained from Table II. For example, the 3-constant theory gives extremely good correlation coefficients, fair slope and intercept values, and acceptable values of average per cent error. Now, if one recognizes that these figures include the scatter in the original data along with any lack of goodness of fit of the original theory, then the picture is even better. Thus, if such a fit is acceptable in a given application, the 3-constant theory can be used to advantage. The results for the 7- and 12-constant theories are better on all counts, and the average per cent error appears to have been brought down to approximately that inherent in the data. As the 12-constant theory does not appear to have a significant advantage over the 7-constant theory, and the latter has a definite superiority over the 3-constant theory, it appears that in general the 7-constant theory should be the most useful.

One should, however, exercise caution in using the constants obtained for various theories to predict stresses for strains below and beyond those used in experimental verification of the applicability. One should, moreover, not expect as good a predic-

TABLE II

COMPARISON BETWEEN MEASURED AND COMPUTED VALUES OF STRESS DIFFERENCES FROM IN VITRO EXPERIMENTS ON 13 DOGS

| Theory | Stress differences | Correlation coefficient | Slope | Intercept | Average absolute per cent error |
|---|---|---|---|---|---|
| | | | | *dynes/cm² × 10³* | |
| 3-Constant theory | $S_\theta - S_r$ | 0.9958 | 0.9308 | 147 | 16.2 |
| | $S_z - S_r$ | 0.9908 | 0.8613 | 161 | 13.0 |
| 7-Constant theory | $S_\theta - S_r$ | 0.9970 | 1.0025 | −9 | 6.7 |
| | $S_z - S_r$ | 0.9960 | 0.9877 | 11 | 4.7 |
| 12-Constant theory | $S_\theta - S_r$ | 0.9975 | 0.9948 | 10 | 4.3 |
| | $S_z - S_r$ | 0.9969 | 0.9827 | 17 | 4.4 |

tion of the incremental moduli at various strains as that obtained for the stress differences, as the incremental moduli involve the derivatives of $W$ with respect to $a$ $b$ and the quality of fit always deteriorates when differentiation is involved. Also, since direct measurements of incremental moduli use experiments involving small perturbations of states of strain, their inherent accuracy is itself limited. If exact measurements of incremental moduli were available, it would be found that the 7- and the 12-constant theories would involve negligible systematic errors while the 3-constant theory would underestimate the moduli in the circumferential and the longitudinal directions. This is indicated by the average slope values given in Table II for the regression lines of the calculated stress differences on the measured stress differences.

Even if the measured values of incremental moduli were available for a given arterial segment, it would probably be more desirable to compute first the constitutive constants of the 7-constant theory and deduce the incremental moduli at chosen states of strain, because in so doing, one properly smooths out the scatter in the original measurements and also at once opens up the possibility of computing incremental moduli at any intermediate chosen state of strain instead of only the states of strain used in the experimentation.

It would be of interest, at this stage, to compare the values of incremental moduli obtained directly with those deduced from the 3-, 7-, and 12-constant theories. For this purpose, all blood vessel segments were also subjected to experimentation to determine incremental moduli over a limited strain range using a method described by Patel et al., 1969. Each segment was held at an essentially constant length by means of a stiff force gauge and subjected to three values of intramural pressure and the experiment repeated at two more constant segment lengths slightly higher and

TABLE III

COMPARISON OF MEASURED AND COMPUTED VALUES OF INCREMENTAL ELASTIC MODULI FROM IN VITRO EXPERIMENTS ON 13 DOGS

| Theory | Incremental moduli | Correlation coefficient | Slope | Intercept | Average absolute per cent error |
|---|---|---|---|---|---|
| | | | | *dynes/cm*$^2$ × *10*$^3$ | |
| 3-Constant theory | $E_\theta$ | 0.9859 | 0.9995 | −511 | 6.0 |
| | $E_z$ | 0.9311 | 0.5147 | 1852 | 28.1 |
| | $E_r$ | 0.9217 | 0.4810 | 1474 | 28.1 |
| 7-Constant theory | $E_\theta$ | 0.9908 | 1.0051 | −560 | 5.8 |
| | $E_z$ | 0.9969 | 1.0101 | 148 | 4.1 |
| | $E_r$ | 0.9779 | 0.9861 | 132 | 5.0 |
| 12-Constant theory | $E_\theta$ | 0.7473 | −0.9700 | −2594 | 31.7 |
| | $E_z$ | 0.9180 | 1.1037 | −2083 | 19.2 |
| | $E_r$ | 0.8781 | 1.0313 | −168 | 13.6 |

lower, in turn, than the first. This provided a set of values of $p$, $F$, $R$, and $L$ at nine closely spaced states of strain, which were sufficient to give one set of values of $E_\theta$, $E_z$, and $E_r$ at the mean state of strain. Also the same data were used to calculate the constitutive constants for the 3-, 7-, and 12-constant theories and the incremental moduli calculated from these constants using equation 45. The 13 sets of measured incremental moduli, $E_\theta$, $E_z$, and $E_r$, were compared with the corresponding computed values by the method previously used in comparing the measured and calculated stress differences. The results are shown in Table III. The values of slopes and intercepts in this case are not of much significance as the rates of change of moduli with strain are not of interest in general. From the values of average per cent error we see that the average error for the 7-constant theory is quite acceptable, but that for the 3- and 12-constant theories is too large; for the 3-constant theory this is an indication of the inadequacy of the theory, but for the 12-constant theory the large errors presumably result from the relative sparseness of data points and the numerical errors associated with computations involving large numbers. It should be mentioned that even for the same arterial segments constitutive constants based on the nine data points (not shown here) obtained using a force gauge were different from those based on the 40 data points obtained using axial weights. This is not surprising. In the latter case the very large strain ranges were covered which tended to give a better over-all picture probably at the expense of the local picture. Moreover, the small creep and relaxation effects could conceivably influence the two types of measurements differently.

To investigate the closeness of correlation between measured and computed in-

TABLE IV

AVERAGE CONSTITUTIVE CONSTANTS (±SEM) FROM IN VITRO EXPERIMENTS ON FOUR DOGS

| Theory | Constitutive constants |
|---|---|
| | *dynes/cm² × 10³* |
| 3-Constant theory | $A_1 = 372 \pm 30$ $B_1 = 219 \pm 19$ $C_1 = 288 \pm 38$ |
| 7-Constant theory | $A_2 = 412 \pm 66$ $B_2 = -67 \pm 131$ $C_2 = 191 \pm 59$ $D_2 = -55 \pm 36$ $E_2 = 162 \pm 51$ $F_2 = 179 \pm 103$ $G_2 = 99 \pm 87$ |
| 12-Constant theory | $A_3 = 1423 \pm 295$ $B_3 = -1235 \pm 432$ $C_3 = -178 \pm 1241$ $D_3 = -1084 \pm 353$ $E_3 = -720 \pm 630$ $F_3 = 2420 \pm 505$ $G_3 = 1496 \pm 2747$ $H_3 = 396 \pm 150$ $I_3 = 395 \pm 131$ $J_3 = 188 \pm 553$ $K_3 = -1138 \pm 218$ $L_3 = -1266 \pm 1769$ |

Minimum $\lambda_\theta = 1.42 \pm 0.03$, maximum $\lambda_\theta = 1.55 \pm 0.03$; minimum $\lambda_z = 1.47 \pm 0.03$; maximum $\lambda_z = 1.51 \pm 0.03$.

TABLE V

COMPARISON BETWEEN MEASURED AND COMPUTED VALUES OF STRESS DIFFERENCES FROM IN VITRO EXPERIMENTS ON FOUR DOGS

| Theory | Stress differences | Correlation coefficient | Slope | Intercept | Average absolute per cent error |
|---|---|---|---|---|---|
| | | | | *dynes/cm² × 10³* | |
| 3-Constant theory | $S_\theta - S_r$ | 0.9975 | 0.9284 | 95 | 12.9 |
| | $S_z - S_r$ | 0.9872 | 0.7299 | 276 | 4.6 |
| 7-Constant theory | $S_\theta - S_r$ | 0.9983 | 0.9968 | 3 | 5.8 |
| | $S_z - S_r$ | 0.9931 | 0.9911 | 8 | 1.9 |
| 12-Constant theory | $S_\theta - S_r$ | 0.9990 | 0.9977 | 2 | 3.4 |
| | $S_z - S_r$ | 0.9783 | 0.9997 | 2 | 0.8 |

cremental moduli over a wider range of stretches, another series of in vitro experiments was conducted. Four mongrel dogs weighing 20.2–37.7 kg (average weight 28.2 kg) were studied in the same manner as the force gauge experiments described above. Because of the limitations of the design of the force gauge, it was possible to vary only the circumferential stretch over a wide range. In the present series the intramural pressure was varied, at a fixed length of approximately 8 cm, over a range of about 30–180 cm $H_2O$ and the experiment was repeated at two other fixed lengths obtained by slightly varying the original length (±1.6 mm). The quantities $p$, $R$, $L$, and $F$ were measured for a total of 38 states of strain. Using the methods described in Patel et al. (1969), the incremental moduli were computed. Using the methods described earlier, the constitutive constants based on the 3-, 7-, and 12-constant theories were calculated. The average values of these constants (±SEM) are given in Table IV. For the 3- and 7-constant theories these constants compare favorably with those in Table I. The small discrepancy between the constants may be attributed to the differences in the ranges of stretches and the difference in the basic nature of the axial weight experiments and force gauge experiments. To examine the goodness of fit of the theories to the data, $S_\theta - S_r$ and $S_z - S_r$ were computed from the constants for each dog and plotted against the measured values. The results shown in Table V show the excellent fit of the 7-

TABLE VI

COMPARISON BETWEEN MEASURED AND COMPUTED VALUES OF INCREMENTAL ELASTIC MODULI FROM IN VITRO EXPERIMENTS ON FOUR DOGS

| Theory | Incremental moduli | Correlation coefficient | Slope | Intercept | Average absolute per cent error |
|---|---|---|---|---|---|
| | | | | *dynes/cm² × 10³* | |
| 3-Constant theory | $E_\theta$ | 0.9600 | 0.9522 | −158 | 10.0 |
| | $E_z$ | 0.8820 | 0.7613 | 106 | 23.2 |
| | $E_r$ | 0.8911 | 0.6064 | 671 | 19.9 |
| 7-Constant theory | $E_\theta$ | 0.9517 | 0.8704 | 494 | 8.1 |
| | $E_z$ | 0.9783 | 0.9957 | 153 | 6.0 |
| | $E_r$ | 0.9151 | 0.8673 | 450 | 10.1 |
| 12-Constant theory | $E_\theta$ | 0.9810 | 0.8730 | 449 | 5.3 |
| | $E_z$ | 0.9861 | 0.8835 | 882 | 5.3 |
| | $E_r$ | 0.9598 | 0.9012 | 336 | 6.5 |

TABLE VII

AVERAGE CONSTITUTIVE CONSTANTS (±SEM) FROM IN VIVO EXPERIMENTS ON 14 DOGS

| Theory | Constitutive constants | | | |
|---|---|---|---|---|
| | *dynes/cm² × 10³* | | | |
| Constant theory | $A_1 = 429 \pm 34$ | $B_1 = 196 \pm 48$ | $C_1 = 379 \pm 56$ | |
| Constant theory | $A_2 = 69 \pm 127$ | $B_2 = 788 \pm 280$ | $C_2 = -204 \pm 180$ | |
| | $D_2 = 179 \pm 120$ | $E_2 = 160 \pm 192$ | $F_2 = -677 \pm 367$ | $G_2 = 693 \pm 225$ |

inimum $\lambda_\theta$ = 1.21 ±0.02; maximum $\lambda_\theta$ = 1.60 ±0.04; minimum $\lambda_z$ = 1.47 ±0.03; maximum $\lambda_z$ 1.53 ± 0.03.

TABLE VIII

COMPARISON BETWEEN MEASURED AND COMPUTED VALUES OF STRESS DIFFERENCES FROM IN VIVO EXPERIMENTS ON 14 DOGS

| Theory | Stress differences | Correlation coefficient | Slope | Intercept | Average absolute per cent error |
|---|---|---|---|---|---|
| | | | | *dynes/cm² × 10³* | |
| 3-Constant theory | $S_\theta - S_r$ | 0.9860 | 0.9934 | 8 | 1.6 |
| | $S_z - S_r$ | 0.9652 | 0.6972 | 329 | 4.1 |
| 7-Constant theory | $S_\theta - S_r$ | 0.9880 | 0.9790 | 28 | 1.2 |
| | $S_z - S_r$ | 0.9796 | 0.9502 | 55 | 1.4 |

TABLE IX

COMPARISON BETWEEN MEASURED AND COMPUTED VALUES OF INCREMENTAL ELASTIC MODULI FROM IN VIVO EXPERIMENTS ON 14 DOGS

| Theory | Incremental moduli | Correlation coefficient | Slope | Intercept | Average absolute per cent error |
|---|---|---|---|---|---|
| | | | | *dynes/cm² × 10³* | |
| 3-Constant theory | $E_\theta$ | 0.9263 | 0.9379 | 72 | 8.1 |
| | $E_z$ | 0.7620 | 0.3107 | 2743 | 34.4 |
| | $E_r$ | 0.7546 | 0.3645 | 1604 | 30.0 |
| 7-Constant theory | $E_\theta$ | 0.9419 | 0.9631 | −280 | 9.4 |
| | $E_z$ | 0.8487 | 0.7930 | 1185 | 6.9 |
| | $E_r$ | 0.9191 | 0.7919 | 244 | 16.0 |
| 12-Constant theory | $E_\theta$ | 0.9144 | 0.8879 | 119 | 10.0 |
| | $E_z$ | 0.7474 | 0.7163 | 1640 | 12.8 |
| | $E_r$ | 0.7748 | 0.7748 | 364 | 15.1 |

and 12-constant theories. Incremental moduli were computed from these constants and compared with those obtained directly by plotting the calculated values against the measured values. The results of comparison are shown in Table VI. Again we see that the 3-constant theory results are only fair but those of the 7-constant and 12-constant theories are good.

The final application of the 3- and 7-constant theories was made to the in vivo data obtained by Patel et al. (1969) for evaluating in vivo incremental moduli. Table VII lists the constitutive constants obtained. It is interesting to see that $A_1$ and $C_1$ from the 3-constant theory are larger than those reported in Table IV for in vitro experiments. This is probably due to the stiffening associated with tethering constraints. The fact that $C_1$ is larger is consistent with the finding of Patel et al. (1969) that the longitudinal incremental modulus was 32% higher in vivo than in vitro. The results of the 7-constant theory are harder to evaluate because of the larger variability of the constants resulting from the limited ranges of $\lambda_\theta$ and $\lambda_z$ used in obtaining the data. Table VIII shows the results obtained by plotting the stress differences calculated from these constants against the measured values. The 7-constant theory is again seen to fit the data extremely well. Finally, the incremental moduli calculated from the theory were plotted against the measured values. Table IX shows the adequacy of the 7-constant theory in this regard.

### *Typical Values of Strain Energy Density*

From the knowledge of the values of the constitutive constants for a given dog, the values of $W$ at any state of strain can be easily computed using equations 16–18. To give an indication of the approximate magnitude of $W$ near physiologic states of strain, values of $W$ were computed for each dog at a state of strain given by $\lambda_\theta = 1.62$ and $\lambda_z = 1.41$ using the constitutive constants of Table I. The average values of $W$ for 13 dogs for the 3-, 7-, and 12-constant theories were found to be $407 \times 10^3$, $354 \times 10^3$, and $376 \times 10^3$ dynes/cm$^2$, respectively. These values compare favorably with the average value 352 ($\pm$26 SEM) $\times 10^3$ dynes/cm$^2$ of $W$ obtained by Patel and Janicki (1970), for the same state of strain, using direct integration of external work done on eight segments of the middle descending thoracic aorta of dogs.

### *Remarks on the Assumption of Thinness of Arterial Wall*

In the preceding development it was assumed that for the purpose of computing constitutive constants the stresses in the arterial wall may be considered to be uniformly

TABLE X

ERRORS IN THE INTRAMURAL PRESSURE ($p$) AND TOTAL LONGITUDINAL FORCE ($F$) AS A RESULT OF ASSUMING THE ARTERIAL WALL TO BE THIN

| Actual $p$ | Average actual $\bar{F}$ | % of error in $p$ | | | % of error in $\bar{F}$ | | | Average $R/h$ |
|---|---|---|---|---|---|---|---|---|
| | | Minimum | Maximum | Average absolute value | Minimum | Maximum | Average absolute value | |
| *cm $H_2O$* | *dynes $\times 10^3$* | | | | | | | |
| 125 | 230 | −3.1 | 11.2 | 5.1 | 1.0 | 8.8 | 4.8 | 12.3 |
| 175 | 295 | −5.3 | 4.8 | 2.4 | −7.0 | 4.2 | 2.6 | 15.2 |

Results based on 13 dogs and constitutive constants of the 7-constant theory from Table I.

distributed. To test the degree of approximation involved we can use the obtained constitutive constants to compute approximate stress distribution through the wall and compute therefrom the internal presure $p$ and the total longitudinal force, $\bar{F} = F + p\pi(R - h/2)^2$, using equations 52 and 53 (Table X). We can then calculate the errors in $p$ and $\bar{F}$ by comparing them with the actual values of $p$ and $\bar{F}$ used. By assuming that the errors in $p$ and $\bar{F}$ are primarily due to errors in the constitutive constants $A$ and $C$, corrected values of these constants can be obtained through an iterative procedure. Such a procedure was carried out for $p$ = 125 cm $H_2O$ and $\bar{F} = 230 \times 10^3$ dynes for the constants of all dogs on which data of Table I are based. It was found that using the corrected values of $A_2$ and $C_2$ as 271 and 235, respectively, and keeping the remaining constants the same, the values of $p$ and $\bar{F}$ checked within 1 %.

## SUMMARY AND CONCLUSION

In conclusion, we have developed and experimentally verified the applicability of three polynomial expressions for the strain energy density function for the orthotropic incompressible, aortic tissue. Whereas all of the three expressions are applicable to various degrees, the third-degree expression with seven constitutive constants is particularly recommended for general use. As one particular application, we are presently using the 7-constant theory to study the progressive changes in the wall properties of veins transplanted in arterial circulation. The importance of such a study in evaluating the results of coronary artery surgery using saphenous vein transplants cannot be overestimated.

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